Linear bound for the dyadic paraproduct on weighted Lebesgue space $L_2(w)$

Oleksandra V. Beznosova*

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Abstract

The dyadic paraproduct is bounded in weighted Lebesgue spaces $L_p(w)$ if and only if the weight w belongs to the Muckenhoupt class A_p^d. However, the sharp bounds on the norm of the dyadic paraproduct are not known even in the simplest $L_2(w)$ case. In this paper we prove the linear bound on the norm of the dyadic paraproduct in the weighted Lebesgue space $L_2(w)$ using Bellman function techniques and extrapolate this result to the $L_p(w)$ case.[1]

1 Introduction

Let D be the collection of dyadic intervals $D = \{I = [k2^{-j}; (k+1)2^{-j}) \mid k, j \in \mathbf{Z}\}$, and let $m_I f$ stand for the average of a locally integrable function f over interval I $m_I f := \frac{1}{|I|}\int_I f$.

The dyadic paraproduct is defined as

$$\pi_b f := \sum_{I \in D} m_I f \; b_I \; h_I$$

where $\{h_I\}_{I \in D}$ is the Haar basis normalized in L_2:

$$h_I(x) = \frac{1}{\sqrt{|I|}} \left(\chi_{I^+}(x) - \chi_{I^-}(x)\right)$$

I^+ and I^- are left and right halves of the dyadic interval I, $b_I := \langle b, h_I \rangle$ where $\langle , \rangle$ stands for the dot product in the unweighted L_2, and b is a locally integrable function.

In order for the paraproduct to be bounded on L_p we need b to be in BMO^d i.e.:

$$\| \, b \, \|_{BMO^d} \; := \; \left(\sup_I \frac{1}{|I|} \int_I |b(x) - m_I b|^2 \, dx \right)^{1/2} \; < \; \infty.$$

We are going to use the fact that the BMO^d norm of b can also be written as:

$$\| \, b \, \|^2_{BMO^d} \; = \; \sup_{J \in D} \frac{1}{|J|} \sum_{I \in D(J)} b_I^2.$$

Paraproducts first appeared in the work of Bony in relation with nonlinear partial differential equations (see [Bo]) and since then took one of the central places in harmonic analysis. Due to the celebrated T(1) theorem of David and Journé [JoDa] a singular integral operator T can be written

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as $T = L + \pi_{b_1} + \pi^*_{b_2}$ where L is almost translation invariant (convolution) operator, ($L1 = 0 = L^*1$), b_1 is the value of T at 1 and $b_2 = T^*(1)$. The dyadic version of this theorem can be found in [Per1]. So, if one is looking for a bound on the norms of some reasonably large class of singular integral operators it is natural to start with the paraproduct and with its simple dyadic "toy" model.

In this paper we prove the linear bound on the norm of dyadic paraproduct on the weighted spaces $L_2(w)$ in terms of the A_2^d characterization of the weight w. And now in order to prove the linear bounds on the norms of operators with standard kernels in the dyadic case one has to concentrate on the operator L.

Paraproduct holds the key to the class of singular integrals with standard kernels. A typical representative of which is the Hilbert transform defined by

$$Hf(x) = P.V.\frac{1}{\pi}\int \frac{f(x)}{x-y}dy.$$

Helson & Szegö in [HeSz] gave necessary and sufficient condition for a weight w so that H maps $L^2(w)$ into itself continuously.

In 1973, Hunt, Muckenhoupt and Wheeden (see [HuMW]) presented a new proof, where for the first time the A_p-condition for the weights appeared as necessary and sufficient condition for the boundedness of the Hilbert transform in $L_p(w)$

$$w \in A_p^d \ \Leftrightarrow \ \| w \|_{A_p} := \sup_{I\in D}\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\right)^{p-1} < \infty.$$

And a year after in [CoFe] Coifman and Fefferman extended this result to a larger class of operators.

The question that has been asked is:

How is the norm of a singular operator in the weighted Lebesque spaces $L_p(w)$ related to the Muckenhoupt (A_p) characteristic of the weight w, $\| w \|_{A_p^d}$. More precisely, what we need is the sharp function $\varphi(x)$ in terms of the growth, such that

$$\| Tf \|_{L_p(w)} \ \le \ C\varphi\left(\| w \|_{A_p^d}\right)\| f \|_{L_p(w)}.$$

This kind of estimates for different singular operators is used a lot in partial differential equations, see [FeKPi], [AISa], [PetVo], [BaJa] and [DrPetVo]. Some partial answers were given to this question.

For the Hilbert transform, Buckley showed power 2 in [Bu], Petermichl and Pott in [PetPo] improved the exponent of $\| w \|_{A_2}$ from 2 to $\frac{3}{2}$ and in 2006 Petermichl got the sharp power 1 for the Hilbert transform, see [Pet1].

Later in [Pet2] Petermichl used similar ideas to show linear bound for the norm of the Riesz transforms.

It was also shown that the norm of the Martingale transform on the weighted space $L_2(w)$ depends linearly on the $\| w \|_{A_2}$, see [Witt].

So now we can claim that singular integral operators related to the above transforms via T(1) theorem admit linear bounds on their norms, i.e. if $T - \pi_{b_1} - \pi^*_{b_2}$ is good enough (one of the operators, for which we know the bound is linear), then

$$\| T \|_{L_2(w)\to L_2(w)} \le C\| w \|_{A_2^d}$$

Boundedness of the paraproduct on the weighted $L_p(w)$ has been known for a long time, a direct proof of it can be found, for example, in [KaPer]. The best known bound on the norm of the dyadic paraproduct so far is

$$\| \pi_b \|_{L_2(w)\to L_2(w)} \le C\phi(\| w \|_{A_2})\| b \|_{BMO^d}$$

with $\phi(x) = x^2$ and it can be found in [DrGPerPet].

First we were able to improve the above result from $\phi(x) = x^2$ to $\phi(x) = x^{3/2}$ without making any significant changes to the structure of the proof. Then using the suggestion of F. Nazarov we tried the duality approach which allowed us to recover 3/2 in multiple ways and using the version of the bilinear embedding theorem from [Pet1] we were able to improve to $\phi(x) = x(1 + \log^{1/2} x)$. Using the sharp version of the bilinear embedding theorem from [NTVo] slightly improved the power of the logarithm in the bound ($\phi(x) = x(1 + \log^{1/4} x)$). And finally, the theorem presented in this paper shows the linear bound and in fact can rely on either one of the bilinear embedding theorems, the one by Nazarov, Treil and Volberg or the one from Petermichl's paper. We would also like to thank S. Treil for a useful conversation.

Let us state the main result now.

Theorem 1. *(Main result) The norm of dyadic paraproduct on the weighted Lebesgue space $L_2(w)$ is bounded from above by a constant multiple of the A_2^d characteristic of the weight w times the BMO^d norm of b, i.e. for all $f \in L_2(w)$ and all $g \in L_2(w^{-1})$*

$$\langle \pi_b f, g \rangle_{L_2} \leq C \parallel w \parallel_{A_2} \parallel b \parallel_{BMO^d} \parallel f \parallel_{L_2(w)} \parallel g \parallel_{L_2(w^{-1})}. \tag{1}$$

Which together with the sharp version of the Rubio De Francia's extrapolation theorem from [DrGPerPet] produces L_p bounds of the following type:

Theorem 2. *Let $w \in A_p^d$ and $b \in BMO^d$. Then the norm of dyadic paraproduct π_b on the weighted $L_p(w)$ is bounded by*

$$\parallel \pi_b \parallel_{L_p(w) \to L_p(w)} \ \leq \ C_1(p) \parallel w \parallel_{A_p^d} \parallel b \parallel_{BMO^d} \quad \text{when } p \geq 2$$

and by

$$\parallel \pi_b \parallel_{L_p(w) \to L_p(w)} \ \leq \ C_2(p) \parallel w \parallel_{A_p^d}^{\frac{1}{p-1}} \parallel b \parallel_{BMO^d} \quad \text{when } p < 2,$$

where $C_1(p)$ and $C_2(p)$ are constants that only depend on p.

This paper is constructed as follows:

Section 2: proof of the main result based on three propositions.

Section 3: Bellman function proof of Proposition 1.

Section 4: Bellman function proof of Proposition 2.

Section 5: Bellman function proof of Proposition 3.

2 Proof of the main result

Proof. In order to prove Theorem 1 it is enough to show that $\forall\ f, g \in L_2$

$$\left\langle \pi_b \left(f w^{-1/2} \right) ; g w^{1/2} \right\rangle \ \leq \ C \parallel w \parallel_{A_2^d} \parallel b \parallel_{BMO^d} \parallel f \parallel_2 \parallel g \parallel_2,$$

where $\left\langle \pi_b \left(f w^{-1/2} \right) ; g w^{1/2} \right\rangle$ can be written as the following sum

$$\left\langle \pi_b \left(f w^{-1/2} \right) ; g w^{1/2} \right\rangle \ = \ \sum_{I \in D} m_I \left(f w^{-1/2} \right) b_I \left\langle g w^{1/2} ; h_I \right\rangle \ =: \ \sum\nolimits_1$$

Now, we are going to decompose this sum into parts using weighted Haar system of functions: Let H_I^w be defined in the following way:

$$H_I^w \ := \ h_I \sqrt{|I|} - A_I^w \chi_I .$$

In order to make $\{H_I^w\}$ an orthogonal system of functions in $L_2(w)$, we take A_I^w to be

$$A_I^w \;:=\; \frac{m_{I^+}w - m_{I^-}w}{2m_I w},$$

then $\{w^{1/2}H_I^w\}$ is orthogonal in L_2 with norms bounded from above by $\| w^{1/2}H_I^w \|_{L_2} \le \sqrt{|I| m_I w}$.

Then by Bessel's inequality we have:

$$\forall g \in L_2 \qquad \sum_{I\in D} \frac{1}{|I| m_I w} \left\langle g; w^{1/2} H_I^w \right\rangle_{L_2}^2 \;\le\; \| g \|_{L_2}^2. \tag{2}$$

The weighted Haar functions were first introduced in [CoJS] and are extremely useful in weighted inequalities, see [NTVo] and [Per2].

We can break $\sum_1$ into two sums:

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{I\in D} m_I\left(fw^{-1/2}\right) b_I \left\langle gw^{1/2}; h_I\right\rangle \\
&= \sum_{I\in D} m_I\left(fw^{-1/2}\right) b_I \frac{1}{\sqrt{|I|}} \left\langle g; w^{1/2}H_I^w\right\rangle + \sum_{I\in D} m_I\left(fw^{-1/2}\right) b_I \frac{1}{\sqrt{|I|}} \left\langle gw^{1/2}; A_I^w \chi_I\right\rangle \\
&=: \sum\nolimits_2 + \sum\nolimits_3 .
\end{aligned}$$

And now we will bound $\sum_2$ and $\sum_3$.

We claim that both sums, $\sum_2$ and $\sum_3$, depend on the $\| w \|_{A_2^d}$ at most linearly:

$$\sum\nolimits_2 \;=\; \sum_{I\in D} m_I\left(fw^{-1/2}\right) b_I \frac{1}{\sqrt{|I|}} \left\langle g; w^{1/2}H_I^w\right\rangle \;\le\; C\| w \|_{A_2^d} \| b \|_{BMO^d} \| f \|_{L_2} \| g \|_{L_2}. \tag{3}$$

and

$$\sum\nolimits_3 = \sum_{I\in D} m_I\left(fw^{-1/2}\right) b_I A_I^w \sqrt{|I|} m_I\left(gw^{1/2}\right) \le C\| w \|_{A_2^d} \| b \|_{BMO^d} \| f \|_{L_2} \| g \|_{L_2}. \tag{4}$$

Before going into the proofs of (3) and (4) let us analyze the partition $\sum_1 = \sum_2 + \sum_3$.

The sum $\sum_2$ is close to the "weighted" version of a paraproduct over a weighted space $L_2(w)$, which behaves similar to the unweighted situation, while $\sum_3$ takes into account the difference between the norm of the paraproduct on weighted and unweighted L_2. In the simplest case $w = const$, $\| w \|_{A_2^d} = 1$, $\sum_1 = \sum_2$, and we recover classical results, while $\sum_3 = 0$.

Note also, that for weights with small A_2^d-characteristics $\sum_2$ will be dominating and $\sum_3$ will be close to 0, while for $\| w \|_{A_2^d}$ large $\sum_3$ becomes more important.

Bound on $\sum_2$ is very straight-forward and very similar to the classical case. We decompose $\sum_2$ into the product of two sums using Cauchy-Schwarz:

$$\begin{aligned}
\sum\nolimits_2 &= \sum_{I\in D} m_I\left(fw^{-1/2}\right) b_I \frac{1}{\sqrt{|I|}} \left\langle g; w^{1/2}H_I^w\right\rangle \\
&\le \left(\sum_{I\in D} m_I^2\left(fw^{-1/2}\right) b_I^2 m_I w\right)^{1/2} \left(\sum_{I\in D} \frac{1}{|I| m_I w} \left\langle g; w^{1/2}H_I^w\right\rangle^2\right)^{1/2}.
\end{aligned}$$

By (2)

$$\sum_{I\in D} \frac{1}{|I| m_I w} \left\langle g; w^{1/2}H_I^w\right\rangle^2 \;\le\; \| g \|_{L_2}^2.$$

So, for (3) it is enough to show that

$$\sum_{I\in D} m_I^2\left(fw^{-1/2}\right) b_I^2 m_I w \;\le\; C\|\,w\,\|^2_{A_2^d}\|\,b\,\|^2_{BMO^d}\|\,f\,\|^2_{L_2}. \tag{5}$$

By the weighted Carleson embedding theorem, which can be found, for example, in [NTVo], and $(d\sigma)$ version of it can be found in [Per2], (5) holds if and only if

$$\forall J\in D\quad \frac{1}{|J|}\sum_{I\in D(J)} m_I^2 w^{-1}\, m_I w\, b_I^2 \;\le\; C\|\,w\,\|^2_{A_2^d}\|\,b\,\|_{BMO^d}\, m_J w^{-1}.$$

And since $\forall I\in D\quad m_I w\, m_I w^{-1}\le \|\,w\,\|_{A_2^d}$, it is enough to verify that

$$\forall J\in D\quad \frac{1}{|J|}\sum_{I\in D(J)} m_I w^{-1}\, b_I^2 \;\le\; C\|\,w\,\|_{A_2^d}\|\,b\,\|_{BMO^d}\, m_J w^{-1}. \tag{6}$$

Inequality (6) follows from the fact that $b\in BMO^d$ and hence the sequence $\{b_I^2\}_{I\in D}$ is a Carleson sequence with Carleson constant $\|\,b\,\|^2_{BMO^d}$:

$$\forall J\in D\quad \frac{1}{|J|}\sum_{I\in D(J)} b_I^2 \;\le\; \|\,b\,\|^2_{BMO^d}, \tag{7}$$

and the following proposition, which we are going to prove using the Bellman function technique in Section 3.

Proposition 1. *Let $w\in A_2^d$ and $\{\lambda_I\}$ be a Carleson sequence of nonnegative numbers*

$$\forall J\in D\quad \frac{1}{|J|}\sum_{I\in D(J)} \lambda_I \;\le\; Q,$$

then $\forall J\in D$

$$\frac{1}{|J|}\sum_{I\in D(J)} \frac{\lambda_I}{m_I w^{-1}} \;\le\; 4Q\, m_J w \tag{8}$$

and

$$\frac{1}{|J|}\sum_{I\in D(J)} m_I w \lambda_I \;\le\; 4Q\|\,w\,\|_{A_2^d}\, m_J w. \tag{9}$$

Estimate (9) applied to $\lambda_I=b_I^2$ and w^{-1} ($w^{-1}\in A_2^d$ and $\|\,w^{-1}\,\|_{A_2^d}=\|\,w\,\|_{A_2^d}$) provides (6), so bound (3) on $\sum_2$ holds.

Now we need to prove bound (4) on $\sum_3$. It is a little bit more involved. We want to show that

$$\sum\nolimits_3 = \sum_{I\in D} b_I A_I^w \sqrt{|I|}\; m_I\left(fw^{-1/2}\right)\; m_I\left(gw^{1/2}\right) \le C\|\,w\,\|_{A_2^d}\|\,b\,\|_{BMO^d}\|\,f\,\|_2\|\,g\,\|_2.$$

We are going to use a version of the bilinear embedding theorem by Petermichl from [Pet1]:

Theorem 3. *(Petermichl) Let $w\in A_2$, $\|\,w\,\|_{A_2}\le Q$. Let $\{\alpha_I\}_{I\in D}$ be a sequence of non-negative numbers such that:*

$$\forall J\in D\quad \frac{1}{|J|}\sum_{I\in D(J)} \alpha_I\, m_I w\, m_I w^{-1} \;\le\; Q,$$

$$\forall J\in D\quad \frac{1}{|J|}\sum_{I\in D(J)} \alpha_I\, m_I w \;\le\; Q m_J w,$$

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} \alpha_I \; m_I w^{-1} \;\leq\; Q m_J w^{-1},$$

then there is a constant $C > 0$ such that $\forall f, g \in L_2$

$$\sum_{I \in D} \alpha_I \; m_I \left(f w^{-1/2}\right) \; m_I \left(g w^{1/2}\right) \;\leq\; CQ \| f \|_{L_2} \| g \|_{L_2}.$$

So, in order to complete the proof it is enough to show that the following three bounds hold:

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} |b_I A_I^w| \sqrt{|I|} \; m_I w \; m_I w^{-1} \;\leq\; C \| w \|_{A_2^d}, \tag{10}$$

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} |b_I A_I^w| \sqrt{|I|} \; m_I w \;\leq\; C \| w \|_{A_2^d} m_J w, \tag{11}$$

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} |b_I A_I^w| \sqrt{|I|} \; m_I w^{-1} \;\leq\; C \| w \|_{A_2^d} m_J w^{-1}, \tag{12}$$

The following Proposition helps us handle the first sum (10).

Proposition 2. *Let w be a weight from A_2^d, then $\forall J \in D$*

$$\frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{m_{I^+} w - m_{I^-} w}{m_I w} \right)^2 |I| m_I^{1/4} w \; m_I^{1/4} w^{-1} \;\leq\; C m_J^{1/4} w \; m_J^{1/4} w^{-1}.$$

Note that a simple consequence of Proposition 2 is

$$\frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{m_{I^+} w - m_{I^-} w}{m_I w} \right)^2 |I| m_I w \; m_I w^{-1} \;\leq\; C \| w \|_{A_2^d}^{3/4} m_J^{1/4} w \; m_J^{1/4} w^{-1}$$

and hence

$$\frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{m_{I^+} w - m_{I^-} w}{m_I w} \right)^2 |I| m_I w \; m_I w^{-1} \;\leq\; C \| w \|_{A_2^d}. \tag{13}$$

Then by Cauchy-Schwarz

$$\frac{1}{|J|} \sum_{I \in D(J)} |b_I A_I^w| \sqrt{|I|} m_I w \, m_I w^{-1} \leq \left(\frac{1}{|J|} \sum_{I \in D(J)} b_I^2 m_I w \, m_I w^{-1} \right)^{\frac{1}{2}} \left(\frac{1}{|J|} \sum_{I \in D(J)} (A_I^w)^2 |I| m_I w \, m_I w^{-1} \right)^{\frac{1}{2}},$$

by (13)

$$\frac{1}{|J|} \sum_{I \in D(J)} (A_I^w)^2 |I| \, m_I w \; m_I w^{-1} \;\leq\; C \| w \|_{A_2^d},$$

and by (7)

$$\frac{1}{|J|} \sum_{I \in D(J)} b_I^2 \, m_I w \; m_I w^{-1} \;\leq\; \| w \|_{A_2^d} \frac{1}{|J|} \sum_{I \in D(J)} b_I^2 \;\leq\; \| w \|_{A_2^d} \| b \|_{BMO^d}^2.$$

Linear bound on the second sum (11) follows by Cauchy-Schwarz, from the sharp result by J.Wittwer [Witt]:

Lemma 1. *(J.Wittwer) Let $w \in A_2^d$ be a weight, then*

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{m_{I_-} w - m_{I_+} w}{2 m_I w} \right)^2 |I| \, m_I w \ \leq \ C \| \, w \, \|_{A_2^d} m_J w$$

and this result is sharp.

and Proposition 1 (inequality (9)) applied to $\lambda_I = b_I^2$:

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} b_I^2 \, m_I w \ \leq \ C \| \, w \, \|_{A_2^d} \, \| \, b \, \|_{BMO^d}^2 \, m_J w.$$

And the next proposition together with (6) allows us to bound the third sum (12) in a similar way.

Proposition 3. *Let w be a weight in A_2^d, then for all dyadic intervals J:*

$$\frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{m_{I^+} w - m_{I^-} w}{m_I w} \right)^2 |I| m_I w^{-1} \ \leq \ C \| \, w \, \|_{A_2^d} m_J w^{-1}.$$

Which completes the proof of the Theorem 1. □

3 Bellman function proof of Proposition 1

We are going to show that for any Carleson sequence $\{\lambda_I\}_{I \in D}$ with constant Q, $\lambda_I \geq 0$

$$\forall J \in D \quad \frac{1}{|J|} \sum_{I \in D(J)} \lambda_I \ \leq \ Q$$

the inequality (8) holds for any dyadic interval J:

$$\frac{1}{|J|} \sum_{I \in D(J)} \frac{\lambda_I}{m_I w^{-1}} \ \leq \ 4 Q m_J w.$$

Note that inequality (9) follows from inequality (8).

Lemma 2. *Suppose there exists a real valued function of 3 variables $B(x) = B(u, v, l)$, whose domain $\mathcal{D}$ is given by those $x = (u, v, l) \in \mathbb{R}^3$ such that*

$$u, \ v, \ l \ \geq \ 0,$$

$$uv \ \geq \ 1,$$

$$l \ \leq \ 1,$$

whose range is given by

$$0 \ \leq \ B(x) \ \leq \ m_J w,$$

and such that the following convexity property holds:

$$\forall x, \ x_\pm \in \mathcal{D} \quad such \ \ that \quad x - \frac{x_+ + x_-}{2} = (0, 0, \alpha)$$

$$B(x) - \frac{B(x_+) + B(x_-)}{2} \ \geq \ \frac{1}{4v} \alpha \tag{14}$$

Then Proposition 1 holds.

Proof of Lemma 2. Fix a dyadic interval J. Let $x_J = (u_J, v_J, l_J)$ where $u_J = m_J w$, $v_J = m_J w^{-1}$ and $l_J = \frac{1}{|J|Q} \sum_{I \in D(J)} \lambda_I$. Clearly for each dyadic J, x_J belongs to the domain $\mathcal{D}$. Let $x_\pm := x_{J^\pm} \in \mathcal{D}$. By definition,

$$x_J - \frac{x_{J^+} + x_{J^-}}{2} \ = \ (0, 0, \alpha_J),$$

where $\alpha_J := \frac{1}{|J|Q} \lambda_J$. Then, by convexity condition (14)

$$\begin{aligned} m_J w \ \geq \ B(x_J) \ & \geq \ \frac{B(x_{J^+})}{2} + \frac{B(x_{J^-})}{2} + \frac{1}{4 v_J} \alpha_J \\ & = \ \frac{B(x_{J^+})}{2} + \frac{B(x_{J^-})}{2} + \frac{1}{4|J|Q m_J w^{-1}} \lambda_J . \end{aligned}$$

Iterating this procedure and using the assumption that $B \geq 0$ on $\mathcal{D}$ we get:

$$m_J w \geq \frac{1}{4|J|Q} \sum_{I \in D(J)} \frac{\lambda_I}{m_I w^{-1}}$$

which implies Proposition 1. $\square$

So, Proposition 1 will hold if we can show existence of the function B of the Bellman type, satisfying the conditions of Lemma 2.

Lemma 3. *The following function*

$$B(u, v, l) := u - \frac{1}{v(1+l)}$$

is defined on $\mathcal{D}$, $0 \leq B(x) \leq u$ for all $x = (u, v, l) \in \mathcal{D}$ and satisfies the following differential inequalities on $\mathcal{D}$:

$$\frac{\partial B}{\partial l} \ \geq \ \frac{1}{4v} \tag{15}$$

and

$$- \, d^2 B \ \geq \ 0. \tag{16}$$

Moreover, conditions (15) and (16) imply the convexity condition (14).

Proof. Range conditions are easy to verify: since all variables are positive on $\mathcal{D}$ and $uv \geq 1$, we have

$$0 \ \leq \ B(u, v, l) = \frac{uv(1+l) - 1}{v(1+l)} = u - \frac{1}{v(1+l)} \ \leq \ u.$$

It is nothing but a calculus exercise to check the differential conditions:

$$\frac{\partial B}{\partial l} \ = \ \frac{1}{v(1+l)^2} \ \geq \ \frac{1}{4v}$$

since $l \geq 1$. And

$$- \, d^2 B \ = \ (du, dv, dl) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{v^3(1+l)} & \frac{1}{v^2(1+l)^2} \\ 0 & \frac{1}{v^2(1+l)^2} & \frac{2}{v(1+l)^3} \end{pmatrix} \begin{pmatrix} du \\ dv \\ dl \end{pmatrix} \ \geq \ 0$$

And finally let us see how differential conditions (15) and (16) imply the convexity condition (14):

$$B(x) - \frac{B(x_+) + B(x_-)}{2} = \left[B(x) - B(\frac{x_+ + x_-}{2}) \right] + \left[B(\frac{x_+ + x_-}{2}) - \frac{B(x_+) + B(x_-)}{2} \right] =$$

$$= \frac{\partial B}{\partial l}(u,v,l')\alpha - \int_{-1}^{1} (1-|t|)b''(t)dt,$$

where $b(t) := B(s(t))$, $s(t) := \frac{1+t}{2}s_+ + \frac{1-t}{2}s_-$, $-1 \le t \le 1$, note that $s(t) \in \mathcal{D}$ whenever s_+ and s_- do since $\mathcal{D}$ is a convex domain. Then differential inequalities trivially imply that $-b''(t) \ge 0$ and

$$B(x) - \frac{B(x_+) + B(x_-)}{2} = \frac{\partial B}{\partial l}(u,v,l')\alpha - \int_{-1}^{1} (1-|t|)b''(t)dt \ge \frac{1}{4v}\alpha.$$

And proofs of both Lemma 3 and Proposition 1 are complete. $\square$

4 Proof of the Proposition 2

We are going to prove that there is a numerical constant $C > 0$, such that for all dyadic intervals $J \in D$

$$\frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{m_{I^+}w - m_{I^-}w}{m_I w} \right)^2 |I| m_I^{1/4} w \, m_I^{1/4} w^{-1} \ \le\ C m_J^{1/4} w \, m_J^{1/4} w^{-1}. \tag{17}$$

using Bellman function technique.

Lemma 4. *Suppose there exists a real-valued function of two variables $B(x) = B(u,v)$, whose domain $\mathcal{D}$ is given by those $x = (u,v) \in \mathbb{R}^2$ such that*

$$u, v \ \ge\ 0 \tag{18}$$

$$uv \ \ge\ 1, \tag{19}$$

whose range is given by

$$0 \ \le\ B(x) \ \le\ \sqrt[4]{uv}, \quad x \in \mathcal{D},$$

and such that the following convexity property holds:

$$\text{if} \ \ x = \frac{x_+ + x_-}{2} \quad \text{then} \quad B(x) - \frac{B(x_+) + B(x_-)}{2} \ \ge\ C \frac{v^{1/4}}{u^{7/4}}(u_+ - u_-)^2 \tag{20}$$

with a numerical constant C independent of everything, then the Proposition 2 will be proved.

Proof. Let $u_I := m_I w$, $v_I := m_I w^{-1}$, $v_+ = v_{I^+}$, $v_- = v_{I^-}$ and similarly for $u_\pm$. Then by Hölder's inequality (u,v) and $(u_\pm, v_\pm)$ belong to the $\mathcal{D}$.

Fix $J \in D$, by the convexity and range conditions

$$\begin{aligned}
|J| \sqrt[4]{m_J w \, m_J w^{-1}} \quad &\ge\quad |J| B(u_J, v_J) \\
&\ge\quad \frac{|J|}{2} B(u_+, v_+) + \frac{|J|}{2} B(u_-, v_-) + |J| C \frac{m_J^{1/4} w^{-1}}{m_J^{7/4} w} (m_{J^+} w - m_{J^-} w)^2 \\
&=\quad |J^+| B(u_+, v_+) + |J^-| B(u_-, v_-) + |J| C \frac{m_J^{1/4} w^{-1}}{m_J^{7/4} w} (m_{J^+} w - m_{J^-} w)^2 .
\end{aligned}$$

Iterating this process and using the fact that $B(u,v) \ge 0$ we get:

$$|J| \sqrt[4]{m_J w \, m_J w^{-1}} \ \ge\ C \sum_{I \in D(J)} |I| \frac{m_I^{1/4} w^{-1}}{m_I^{7/4} w} (m_{J^+} w - m_{J^-} w)^2,$$

which completes the proof of Lemma 4. $\square$

Now, in order to complete the proof of (17) we need to show existence of the Bellman type function B which satisfies the conditions of Lemma 4.

Lemma 5. *The following function*

$$B(u,v) := \sqrt[4]{uv}$$

is defined on $\mathcal{D}$, $0 \le B(u,v) \le \sqrt[4]{uv}$ for all $(u,v) \in \mathcal{D}$, and satisfies the following differential inequality in $\mathcal{D}$:

$$-d^2B \;\ge\; \frac{1}{8}\frac{v^{1/4}}{u^{7/4}}|du|^2. \tag{21}$$

Furthermore, this implies the convexity condition (20) of Lemma 4.

Proof. Since u and v are positive in the domain $\mathcal{D}$, function $B = \sqrt[4]{uv}$ is well defined on $\mathcal{D}$ and condition $0 \le B(u,v) \le \sqrt[4]{uv}$ is trivially satisfied.

Let us prove the differential inequality (21) now:

$$-d^2B \;=\; \frac{1}{16}(du,dv)\begin{pmatrix} 3v^{\frac14}u^{\frac{-7}{4}} & -v^{\frac{-3}{4}}u^{\frac{-3}{4}} \\ -v^{\frac{-3}{4}}u^{\frac{-3}{4}} & 3v^{\frac{-7}{4}}u^{\frac14}\end{pmatrix}\begin{pmatrix} du \\ dv\end{pmatrix}$$

$$= \frac{1}{8}(du,dv)\begin{pmatrix} v^{\frac14}u^{\frac{-7}{4}} & 0 \\ 0 & v^{\frac{-7}{4}}u^{\frac14}\end{pmatrix}\begin{pmatrix} du \\ dv\end{pmatrix} + \frac{1}{16}(du,dv)\begin{pmatrix} v^{\frac14}u^{\frac{-7}{4}} & -v^{\frac{-3}{4}}u^{\frac{-3}{4}} \\ -v^{\frac{-3}{4}}u^{\frac{-3}{4}} & v^{\frac{-7}{4}}u^{\frac14}\end{pmatrix}\begin{pmatrix} du \\ dv\end{pmatrix}$$

$$\ge\; \frac{1}{8}v^{\frac14}u^{\frac{-7}{4}}|du|^2,$$

as we wanted to show.

Now we only need to check the convexity condition (20). We fix an interval I and let

$$b(t) \;:=\; B(u_t,v_t), \quad -1 \;\le\; t \;\le\; 1,$$

where

$$u_t \;:=\; \frac{1}{2}(t+1)u_+ + \frac{1}{2}(1-t)u_-$$

and

$$v_t \;:=\; \frac{1}{2}(t+1)v_+ + \frac{1}{2}(1-t)v_-.$$

What we want to show is

$$b(0) - \frac{b(1)+b(-1)}{2} \;\ge\; C\frac{v^{1/4}}{u^{7/4}}|du|^2.$$

It is easy to see that

$$b(0) - \frac{1}{2}\left(b(-1)+b(1)\right) \;=\; -\frac{1}{2}\int_{-1}^{1}(1+|t|)b''(t)dt.$$

Note that

$$-b''(t) \;\ge\; \frac{1}{32}v_t^{1/4}u_t^{-7/4}\left(u_1-u_{-1}\right)^2 \tag{22}$$

and that $\forall t \in [-1/2;1/2]$

$$u_t \;=\; u_0 + \frac{1}{2}t(u_1-u_{-1}),$$

since domain $\mathcal{D}$ is convex $u_t \in \mathcal{D}$, and

$$|u_1-u_{-1}| \;\le\; |u_1|+|u_{-1}|, \quad |t| \le 1/2, \;\; u_1,u_{-1} \ge 0,$$

$$-u_0 \;=\; -\frac{1}{2}(u_1+u_{-1}) \;\le\; t(u_1-u_{-1}) \;\le\; \frac{1}{2}(u_1+u_{-1}) \;=\; u_0,$$

so $u_t \ \le\ \frac{3}{2}u_0$ and similarly $v_t \ \ge\ \frac{1}{2}v_0$ for $t \in [-1/2; 1/2]$. Together with (22) it makes

$$-b''(t) \ \ge\ C v_0^{1/4} u_0^{-7/4}(u_1 - u_{-1})^2.$$

So,

$$B(u,v) - \frac{1}{2}(B(u_+,v_+) - B(u_-,v_-)) \ =\ b(0) - \frac{1}{2}(b(1) + b(-1)) \ \ge\ C\frac{v^{1/4}}{u^{7/4}}|du|^2$$

with numerical constant C independent of everything. Which completes the proof of Lemma 5 and Proposition 2. □

5 Proof of the Proposition 3

First note that since for every dyadic interval I we have $m_I w\ m_I w^{-1} \le \| w \|_{A_2^d}$, it is enough to show that

$$\forall J \in D \quad \frac{1}{|J|}\sum_{I \in D(J)} \frac{(m_{I^+}w - m_{I^-}w)^2}{m_I^3 w}|I| \ \le\ C m_J w^{-1} \tag{23}$$

for some numerical constant C.

Lemma 6. *Suppose there exists a real-valued function of two variables $B(x) = B(u,v)$, whose domain $\mathcal{D}$ is given by those $x = (u,v) \in \mathbb{R}^2$ such that*

$$u, v \ \ge\ 0, \tag{24}$$

$$uv \ \ge\ 1, \tag{25}$$

whose range is given by

$$0 \ \le\ B(x) \ \le\ v$$

and such that the following convexity property holds:

$$\text{if} \quad x = \frac{x_+ + x_-}{2} \quad \text{then} \quad B(x) - \frac{B(x_+) + B(x_-)}{2} \ \ge\ C\frac{1}{u^3}(u_+ - u_-)^2 \tag{26}$$

with some numerical constant C independent of everything. Then Proposition 3 will be proved (inequality (23) holds for all dyadic intervals J).

Proof. Let $u_I := m_I w$, $v_I := m_I w^{-1}$, $v_+ = v_{I^+}$, $v_- = v_{I^-}$ and similarly for $u_\pm$. Then by Hölder's inequality (u,v) and $(u_\pm, v_\pm)$ belong to the $\mathcal{D}$.

Fix $J \in D$, by the convexity property and range conditions

$$\begin{aligned}
|J| m_J w^{-1} \quad &\ge \quad |J| B(u_J, v_J) \\
&\ge \quad \frac{|J|}{2}B(u_+,v_+) + \frac{|J|}{2}B(u_-,v_-) + C|J|\frac{1}{m_J^3 w}(m_{J^+}w - m_{J^-}w)^2 \\
&= \quad |J^+|B(u_+,v_+) + |J^-|B(u_-,v_-) + C|J|\frac{1}{m_J^3 w}(m_{J^+}w - m_{J^-}w)^2.
\end{aligned}$$

Iterating this process and using positivity of function B, we get

$$|J| m_J w^{-1} \ \ge\ C \sum_{I \in D(J)} |I| \frac{1}{m_I^3 w}(m_{I^+}w - m_{I^-}w)^2,$$

which completes the proof of Lemma 6. □

To prove inequality (23) and Proposition 3 we need to show the existence of the function B of the Bellman type satisfying conditions of Lemma 6.

Lemma 7. *The following function*

$$B(u,v) \;=\; v - \frac{1}{u}$$

defined on domain $\mathcal{D}$, $0 \le B(u,v) \le v$ for all $(u,v) \in \mathcal{D}$ and satisfies the following differential inequality in $\mathcal{D}$:

$$-d^2 B \;\ge\; \frac{2}{u^3}|du|^2.$$

Moreover, it implies the convexity condition (26) with some numerical constant C independent of everything.

Proof. First note that since $uv \ge 1$ and u and v are both positive in the domain $\mathcal{D}$, B is well-defined and

$$0 \;\le\; B(u,v) \;=\; \frac{uv-1}{u} \;=\; v - \frac{1}{u} \;\le\; v$$

on D and $-d^2B \;=\; 2u^{-3}|du|^2$.

Convexity condition (26) follows from this in practically the same way as in Proposition 2. □

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Oleksandra V. Beznosova
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131, USA
alexbeznosova@yahoo.com